SPS 2349 Test of Hypothesis
Generalized Likelihood Ratio Test

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Introduction

Suppose a random variable X has a pdf $f(x_1, \theta_1, \theta_2, ..., \theta_k)$ depending on the unknown parameters $\theta_1, \theta_1, ..., \theta_k$

The set of all parameters is denoted by Ω known as the parameter space.

Let Ω_0 be a subset of Ω consider the hypothesis:

$$H_0 : (\theta_1, \theta_2, ..., \theta_k) \in \Omega_0 \ \ vs \ \ H_a : (\theta_1, \theta_2, ..., \theta_k) \in \Omega - \Omega_0$$

The above case is a composite hypothesis against a composite one. In this case we do not use Neyman Pearson lemma but we use a more generalized form of it generalized likelihood ratio test(LRT)

We denote the likelihood function of a sample $x_1, x_2, ..., x_n$ by

$$L(\Omega) = \prod_{i=1}^{n} f(x_i, \theta_1, \theta_2, ..., \theta_k)$$

Let $L(\hat{\Omega}) = \max L(\Omega)$ be maximized $L(\Omega)$ and $L(\Omega_0) = \max(L(\Omega_0))$ be maximized $L(\Omega_0)$ with Ω_0 only

We take the likelihood ratio as a quotient

$$\lambda = \frac{L(\hat{\Omega}_0)}{L(\hat{\Omega})}$$

Since λ is the quotient of non-negative functions it is greater than or equal to zero. Further since $\Omega_0 \in \Omega$ then

$$L(\hat{\Omega}) \leq L(\hat{\Omega})$$

hence $0 \leq \lambda \leq 1$

To test $H_0 : \theta \in \Omega_0$ against $H_a : \theta \in \Omega - \Omega_0$ the critical region for the likelihood ratio test is the set of points in the sample for which

$$\lambda = \frac{L(\hat{\Omega}_0)}{L(\hat{\Omega})} \leq k$$

Where k is a constant determined so that the level of significance is α

$$P\Big[\lambda \leq k | H_0 \ \ is \ \ true\Big] = \alpha$$

Example 1

$X \sim N(\mu, \sigma^2)$ where $\mu \& \sigma^2$ are unknown. Test the hypothesis

$$H_0 : \mu = \mu_0 \ \ vs \ \ H_1 : \mu \neq \mu_0$$

Solution

Let $x_1, x_2, ..., x_n$ be a random sample of size n from X then

$$L(X, \mu, \sigma^2) = \left(\frac{1}{2\pi\sigma^2}\right)^{\frac{n}{2}} e^{-\frac{1}{2\sigma^2} \sum_{i=1}^{n}(x_i - \mu)^2}$$

In

$$\Omega = \{(\mu, \sigma) : -\infty < \mu < \infty, \sigma > 0\}$$
$$\Omega_0 = \{(\mu, \sigma) : \mu = \mu_0, \sigma > 0\}$$

$L(\hat{\Omega}) = \max_{\mu, \sigma^2 \in \Omega} L(X, \mu, \sigma^2)$

$$\ln L(\mu, \sigma^2) = -\frac{n}{2} \ln 2\pi - \frac{n}{2} \ln \sigma^2 - \frac{1}{2\sigma^2} \sum_{i=1}^{n}(x_i - \mu)^2$$

$$\frac{\partial l}{\partial \mu} = 0 \rightarrow 2 \sum_{i=1}^{n} \frac{(x_i - \mu)}{2\sigma^2} 0 \rightarrow \hat{\mu} = \bar{x}$$

$$\frac{\partial \ln L}{\partial \sigma^2} = 0 \rightarrow -(\frac{n}{2})\frac{1}{\sigma^2} + \frac{\sum_{i=1}^{2}(x_i - \mu)^2}{2(\sigma^2)^2}$$

$$L(\hat{\Omega}) = \left(\frac{1}{2\pi\hat{\sigma^2}}\right) e^{-\frac{1}{2\sigma^2} \sum_{i=1}^{n}(x_i - \bar{x})^2}$$

$$= \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \left[\frac{n}{\sum_{i=1}^{n}(x_i - \bar{x})^2}\right]^{\frac{n}{2}} e^{-\frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2}} \qquad (1)$$

$$= \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \left[\frac{n}{\sum(x_i - \bar{x})^2}\right]^{\frac{n}{2}} e^{-\frac{n}{2}}$$

Under Ω_0 the maximum likelihood estimates $\mu \& \sigma^2$ are

$$\hat{\mu} = \mu_0, \hat{\sigma^2} = \frac{1}{n} \sum_{i=1}^{n}(x_i - \mu_0)^2$$

$$L(\hat{\Omega}) = \left(\frac{1}{2\pi\hat{\sigma^2}}\right) e^{-\frac{1}{2\sigma^2} \sum_{i=1}^{n}(x_i - \mu_0)^2}$$

$$= \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \left[\frac{n}{\sum_{i=1}^{n}(x_i - \mu_0)^2}\right]^{\frac{n}{2}} e^{-\frac{\sum_{i=1}^{n}(x_i - \mu_0)^2}{\frac{1}{n}\sum_{i=1}^{n}(x_i - \mu_0)^2}} \qquad (2)$$

$$= \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \left[\frac{n}{\sum(x_i - \mu_0)^2}\right]^{\frac{n}{2}} e^{-\frac{n}{2}}$$

Example 1 INTRODUCTION

Therefore the likelihood ratio criterion is given by

$$\lambda = \frac{L(\hat{\Omega}_0)}{L(\hat{\Omega})} \leq k$$

$$= \frac{\left[\frac{n}{\sum_{i=1}^n (x_i - \mu_0)^2}\right]^{\frac{n}{2}}}{\left[\frac{n}{\sum_{i=1}^n (x_i - \bar{x})^2}\right]^{\frac{n}{2}}} \leq k \tag{3}$$

$$= \left[\frac{\sum_{i=1}^n (x_i - \bar{x})^2}{\sum_{i=1}^n (x_i - \mu_0)^2}\right]^{\frac{n}{2}} \leq k$$

The test is to reject H_0 when $\lambda \leq c$ Where c is such that

$$P[\lambda \leq c] = \alpha$$

$$\sum_{i=1}^n (x_i - \mu_0)^2 = \sum_{i=1}^n (x_i - \bar{x} + \bar{x} - \mu_0)^2$$

$$= \sum (x_i - \bar{x})^2 + 2(\bar{x} - \mu_0) \sum (x_i - \bar{x}) + n(\bar{x} - \mu_0)^2 \tag{4}$$

$$= \sum (X_i - \bar{X})^2 + n(\bar{x} - \mu_0)^2$$

So

$$\lambda = \left[\frac{\sum (x_i - \bar{x})^2}{\sum (x_i - \bar{x})^2 + n(\bar{x} - \mu_0)^2}\right]^{\frac{n}{2}}$$

$$\lambda = \left[\frac{1}{1 + \frac{\sum (\bar{x} - \mu_0)^2}{\sum (x_i - \bar{x})^2}}\right]^{\frac{n}{2}}$$

$$T = \frac{\sqrt{n}(\bar{x} - \mu_0)/\sigma}{\left[\frac{\sum (x_i - \bar{x})^2}{\sigma^2 (n-1)}\right]^{\frac{1}{2}}} = \frac{\sqrt{n}(\bar{x} - \mu_0)}{\sqrt{\frac{\sum (x_i - \bar{x})^2}{n-1}}}$$

This variable has the student t distribution with $n - 1$ degrees of freedom when H_0 is true.

Thus

$$T^2 = \frac{n(\bar{x} - \mu_0)^2}{\frac{\sum (x_i - \bar{x})^2}{n-1}}$$

$$\lambda = \left[\frac{1}{1 + \frac{T^2}{n-1}}\right]^{\frac{n}{2}}$$

$$\lambda \leq c \to T^2 \geq k$$

We therefore look for a constant c such that

$$P[\lambda \leq c | H_0] = \alpha = P[T^2 \geq k | H_0]$$

Example 2 *INTRODUCTION*

It is enough to find a constant k such that

$$P[T^2 \geq k|H_0] = \alpha = P[|T| > \sqrt{k}|H_0] = \alpha$$

or

$$P[-k < T < k] = 1 - \alpha$$

Since t distribution is symmetric about zero we usually find k such that

$$P[T > \sqrt{k}] = \frac{\alpha}{2} = P[T < -\sqrt{k}]$$

So the test is to compute the statistic

$$T = \frac{\sqrt{n}(\bar{x} - \mu_0)}{\sqrt{\frac{\sum (x_i - \bar{x})^2}{n-1}}}$$

And reject H_0 if $|T| \geq t_{1-\frac{\alpha}{2}(n-1)}$

Example 2

Alfafa yields of six test plots are given by

$$1.5, 1.9, 1.2, 1.4, 2.3, 1.3$$

per acre. Use a critical region of 0.05 to test

$$H_0 : \mu = 1.8 \ \ vs \ \ \mu \neq 18$$

Assume the yields have a normal distribution of mean μ

Solution

The test is to compute

$$T = \frac{\sqrt{n}(\bar{x} - \mu_0)^2}{\sqrt{\frac{(x_i - \bar{x})^2}{n-1}}}$$

and reject H_0 when $|T| > c$ Where c is such that

$$P[|T| > c|H_0] = \alpha$$

or

$$P[-c < T < c|H_0] = 1 - \alpha = 0.05$$

Here

$$n = 6$$

and

$$\bar{x} = 1.6$$

and

$$\sum (x_i - \bar{x})^2 = 0.88$$

Therefore

$$T = \frac{\sqrt{n}(\bar{x} - \mu_0)}{\sqrt{\frac{\sum(x_i-\bar{x})^2}{n-1}}} = \frac{\sqrt{6}(1.6 - 1.8)}{\sqrt{\frac{0.88}{5}}} = -1.17$$

From the tables with 5 df the value of c is 2.571.

Since

$$|T| = 1.17$$

and

$$1.17 < 2.571$$

we fail to reject H_0 at 0.05 level of significance.

Example 3

$$X \sim N(\mu, \sigma^2)$$

To test

$$H_0 : \mu = \mu_0, \sigma = \sigma_0 \ \ is \ \ known$$

vs

$$H_1 : \mu = \mu_1$$

Solution

$$\Omega = \{(\mu, \sigma) : -\infty < \mu < \infty, \sigma = \sigma_0\}$$

$$\Omega_0 = \{(\mu, \sigma) : \mu = \mu_0, \sigma = \sigma_0\}$$

$$L(X, \mu, \sigma) = \left(\frac{1}{2\pi\sigma_0^2}\right)^{\frac{n}{2}} e^{-\frac{1}{2\sigma_0^2} \sum(x_i-\mu)^2}$$

$$L(\hat{\Omega}) = \left(\frac{1}{2\pi\sigma_0^2}\right)^{\frac{n}{2}} e^{-\frac{1}{2\sigma_0^2} \sum(x_i-\bar{x})^2}$$

$$L(\hat{\Omega}) = \left(\frac{1}{2\pi\sigma_0^2}\right)^{\frac{n}{2}} e^{-\frac{1}{2\sigma_0^2} \sum(x_i-\mu_0)^2}$$

Hence

$$\lambda = \frac{L(\hat{\Omega_0})}{L(\hat{\Omega})} = \frac{e^{-\frac{1}{2\sigma_0^2} \sum(x_i-\mu_0)^2}}{e^{-\frac{1}{2\sigma_0^2} \sum(x_i-\bar{x})^2}} = e^{-\frac{1}{2\sigma_0^2} \sum(x_i-\mu_0)^2 + \frac{1}{2\sigma_0^2} \sum(x_i-\bar{x})^2}$$

The critical region is